The Design of Everyday Numbers

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What are numbers? Why are they everywhere? Why do they work so well? Can they be made better? Here, we will see where numbers come from and how to approach these questions.

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1. Introduction

I am sure we would all agree that the wheel is a crucial invention that has pushed humanity forward. Hence, it is important to make sure that it is well designed. Wheels come in many different forms -- from simple blocks of wood in a toy train to sophisticated multifeatured ones in a racing car -- depending on various factors such as need, aesthetic, technology and cost. However, they all have the same underlying structure of a circular disc. Could this seemingly perfect circular disc design be improved? How should one go about customising wheels for a particular object? To answer these questions adequately and possibly reinvent the wheel, one would need to understand the underlying principles of wheels and their design.

Similarly, numbers are very influential. They seem to work smoothly, though we should question whether they are the right tools to have. Likewise, various systems of numbers have been developed. We have a few standard systems -- from the natural numbers children first learn to count with, to the complex numbers used by physicists and engineers. There are also many lesser known systems, such as the octonions and the ordinal numbers. Moreover, each of these number systems has countless customisations that suit different purposes. In this article, we will try to understand where numbers come from, with a focus on the most ubiquitous systems: the natural numbers (with numbers like 1, 2 and 3) and the real numbers (with numbers like 1, -0.5 and π).

2. Comparing Apples and Oranges

We need some sense of what "numbers" mean in order to analyse them. We can begin with a glance at the concept of counting, which will shed light on this question and allow us to proceed with our investigation.

The phrase "comparing apples and oranges" expresses the idea that objects need to be sufficiently similar for meaningful comparisons. Contrary to this popular idiom, although the two fruits differ in many aspects, they certainly share the property of being countable. For example:

- "Three apples are more than two apples" is analogous to "Three oranges are more than two oranges."
- "Putting three apples and two apples together, we have five apples" is analogous to "Putting three oranges and two oranges together, we have five oranges."

Counting apples is equivalent to counting oranges: a person who can count apples would also be able to count oranges, and vice versa. In fact, counting has little to do with apples or oranges. Many types of objects can be counted: apples, oranges, fingers, days, dragons, etc. are all counted in an analogous way. By drawing this analogy and extracting the common underlying structure, we arrive at a system of "pure" counting with entities like "one", "two", "plus", "greater than", and so on (Figure 1). We can then make sense of sentences like "Three plus two is greater than four",and reapply this knowledge to apples and oranges.

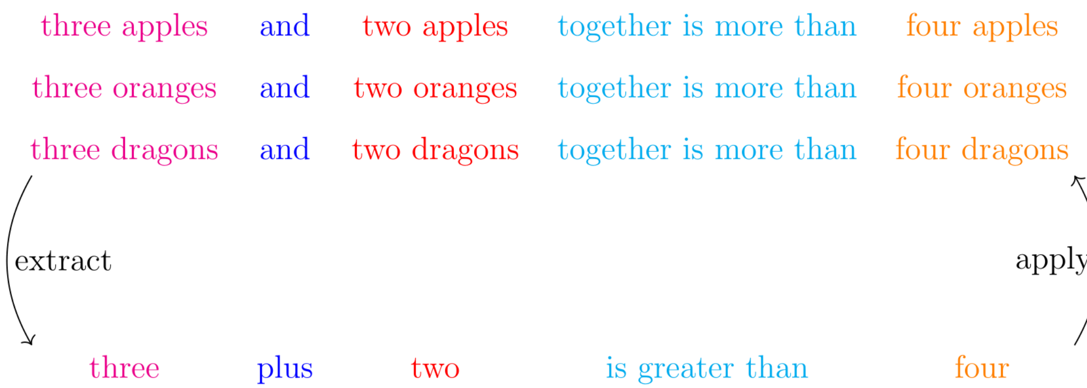

Figure 1 - An Analogy of Counting

In general, a number system extracts from certain concepts, which can then be studied abstractly and applied wherever these concepts are found.

Relating to number systems, there are two main directions of analysis: system-oriented and concept-oriented. In a system-oriented approach, we investigate a number system by inspecting its extractions, applications, and mathematical properties, asking questions like the following.

- ❖ What are some primary concepts that the system is extracted from? In general, a number system can be extracted from many concepts and it would be difficult to list them all. However, a system is usually associated with a few prototypical ones. For example, the natural numbers are associated with counting.

- What modifications to the system are available? How does the system compare to other systems? A number system can be tailored for specific usage or similar concepts (e.g. the natural numbers can be modified to count days of the week, which are limited to seven). This also indicates the flexibility, capabilities and limitations of the system.
- How are the entities in the system represented? Representations are necessary for analysis, computation, recording and communication. Effective representations are important for efficient and intuitive usage.
- What is the collection of axioms (i.e. rules) that characterises the system? Having a collection of axioms is essential, since:
    - To know whether the system can be extracted from a concept, we only need to check the concept against a collection of axioms.
    - Axioms allow us to describe the system directly, which is particularly important for teaching and implementations in computers.
    - It is convenient to create modifications and extensions by dropping, adding and changing axioms.
    - Axioms prevent mistakes since they tell us for what we are allowed to do when using the system.
    - From the axiom, we can derive handy tricks and facts for more efficient usage.

On the other hand, in a concept-oriented approach, we are interested in the systems that best formalise a given concept.

- In what ways can the concept be formalised? A concept can have many facets and there can be multiple formalisations.
- What assumptions are being made in a particular formalisation? Assumptions might not completely reflect our perceptions of the concept, but idealisations are

often necessary for useful and insightful formalisations. Awareness of assumptions is critical for recognising potential flaws.

- What formal features are available to the numbers? For example, the natural numbers have features such as comparison, addition and multiplication. Without any features, these numbers are just names (like 'Matthew') that we cannot do very much with and the system would not be very useful:
  - What notion in our concept would the given feature extract from? A feature would be more relevant if it had a corresponding notion. For example, addition corresponds to putting two collections together. However, an "unnatural" feature might also be powerful and provide new insights about our concept.
  - Can the given feature be amended and in what ways? A certain feature might be desirable, though it might conflict with other features or there could be multiple ways of amendment.
  - What are the utilities in amending or not amending the given feature? Amending a feature can be useful, but can also create complications. For example, addition is often useful, but unnecessary for a tally counter which only lists the numbers in order.
- Can we develop a widely applicable standard system? Given the many choices of features, various systems are possible, which can cause confusion for potential users. Hence, a small number of standardised systems would be desirable. We have a few guidelines for choosing these:
  - It is better to have a tool that is left aside than to not have the tool when it is needed. A standardised system should be inclusive enough to meet a wide range of needs. However, we should also be choosy: conflicts can occur, and simplicity is beneficial.
  - Something often used or requested should be included, if possible.

  - The system should be flexible. Ideally, users can obtain common alternative systems through easy adjustments to the standardised ones.
- What are the properties of a given formalisation? How can the system be efficiently represented and used? This is where the system-oriented analysis comes in.

These two approaches will guide us throughout this article.

3. Systems for Counting

We shall now have a more detailed look at counting within the framework of a concept-oriented analysis. We have a type of object (e.g. apples) which has a smallest unit (e.g. an apple) that we would like to count. For any collection of such objects, we want to assign its "quantity" to a "number" in our system (e.g. "Three" represents "a collection of three apples".) There are many assumptions to be made. We will list a few important and perhaps subtle ones.

- We suppose that it is clear what the smallest unit of counting is. It is perhaps possible to use our system to count apples that are sliced, burnt, rotten, blended, etc., though we leave it for the user to decide on an appropriate unit of counting in each of these contexts.
- We assume that, for example, we can have a collection of just one apple. This might not be the case if, for example, we are counting trees in forests, which would necessarily have more than an individual (since a forest has multiple trees).

- We expect that, for example, we can always add an apple to any given collection of apples to get another collection. This might not be the case if, for example, we assumed that the apples are to be put into a basket of a fixed size.
- We expect that, for example, we cannot add a few apples to a collection of apples and end up with the same number of apples. This might not be the case if, for example, we naively try to count blended apples: two apples blended together could be the same as just one big blended apple.

We will now look at a few features available to the numbers. For each feature, we will address the three questions as specified above.

- Succession.
  - Each number can have a successor: the next number after it. Taking the successor corresponds to adding an object to a collection (Figure 2).

Figure 2 - Succession corresponds to adding an object.

  - There is only one way to include the feature of succession -- the assumptions specified above give restrictions as to how succession should behave. For example, we expect to get a different size by adding an object to a collection, so we might have an axiom that says that the successor of a number should not be itself (Figure 3).

	Adding an apple to	a collection	gives	a different size	
extract ↓					↑ apply
	The successor of	a number	is	not itself	

Figure 3 – Rules for succession come from our assumptions.

- Succession is an essential feature. Many other features are built on top of it. It is useful and easy to understand -- it is the feature that children first learn about.

- Addition.
  - It is quite clear what addition corresponds to: the size given from putting two collections together.
  - Addition can be defined by using successors. In fact, it is an extension of succession, which is essentially "adding by one".
  - Addition is fundamental to the applicability of the number system. It allows us to determine the size of a collection by partitioning it into multiple sub-collections and summing their sizes. On the other hand, there is utility in a system with succession but without addition, such as the use of a tally counter for tracking arrivals.
- Multiplication.
  - We certainly use multiplication to count in practice. However, it is not always clear what multiplication corresponds to -- what should two apples "times" three apples mean? To have multiplication, we actually need a more involved extraction process, which we will look at later in this article.
  - Multiplication can be defined with successors.
  - Similar to addition, multiplication is fundamental, though it can be complicated.

- Zero.
  - There is little ambiguity as to what zero corresponds to: an empty collection.
  - Zero behaves well with standard features (e.g. succession, addition, multiplication, etc.).
  - Having zero is often useful or even necessary. For example, zero (or something synonymous) would appear in the record for a student who never attended any lectures. On the other hand, it can also be useful to teach without zero and there are many instances where zero is irrelevant, such as when we are not interested in keeping track of empty collections.
- Negative numbers.
  - It is fairly clear what negative numbers correspond to: negative two represents "missing two objects".
  - As with zero, negative numbers can be included with no issues.
  - The motivation for negative numbers is similar to that for zero, though the utility for not having negative numbers is more apparent. For example, it would be quite tricky for an attendance record to have a negative number for a student.
- Infiniteness.
  - The size of an "unlimited" collection would be infinite.
  - We will not go into details regarding the notion of infinity. What should be noted is that infinity can be incorporated in multiple ways.
  - On one hand, features related to infinity are often avoided or not used and make the system more complicated. On the other hand, infinity is a very practical and indispensable notion, as used in disciplines such as physics and computer science.

One possible choice for a standardised number system would be to include only succession, which is the most ubiquitous feature across different contexts. The succession system is simple and contains the necessary functionality for building other common concepts. However, counting often involves much more than succession. The natural numbers with their usual operations would also be a good proposal. Addition and multiplication have clear meanings for counting and constitute a more well-rounded toolkit. They are built from succession, so they are harmless to include.

Remark: One might question whether "natural numbers" should refer to a system with or without zero. This is an important question to resolve, at the very least to avoid miscommunication. However, this is largely a matter of convention and the question is left to interested readers.

Having decided upon the features that we want to include, we should think from a system-oriented perspective about how they can be effectively used for counting. We will only make a few brief remarks.

- It should be noted that the system of natural numbers is very flexible. Many other counting-related systems -- trees in forests, apples in a basket, missing items, infinity -- can all be built from the natural numbers.
- The representation of natural numbers is an important topic what we will come back to later.
- The assumptions we have made regarding counting should guide us in our choice of axioms, though we will not go into details here.

- As mentioned above, the natural numbers can be built from the succession system. This has two important implications:
  - Extraction of the natural numbers can be broken down into two steps. Firstly, we need to ensure that the target concept (e.g. counting) does indeed instantiate succession as specified by the axioms for the succession system. Secondly, we need to make sure that other features, like addition and multiplication, built from succession, do in fact correspond to the notions that we anticipate.
  - Wherever we find the succession system, we can apply the extra features from the natural numbers. This might lead to surprising, and possibly unintended, results. For example, we can "multiply" apples, though it is not clear what this means; this issue is addressed in the next section.

4. One Number, Two Systems

Let us try to extract numbers from the positions in a race. We use "1", "2", "3", etc. to respectively represent "first in the race", "second in the race", "third in the race", etc. We then come to the succession system: the first is followed by the second, which is followed by the third, and so on.

Suppose I finish three places after the second place. What position in the race am I at? By using addition borrowed from the natural numbers, we have that 3+2=5. It is tempting to conclude that I came fifth in the race. The answer is correct, though the reasoning is flawed.

- In the expression "3+2", we were trying to use "3" to represent "third after" though we actually meant for "3" to represent "third in the race".
- How should we interpret "third in the race" plus "second in the race"? Knowing who are third and second in the race generally tell us little about the fifth place, which might not even exist!

The issue is that "3" is confusingly being used to represent multiple things simultaneously: "third in the race" and "third after". In fact, we are working in a system with two subsystems: a succession system that represents positions in a race and an addition system that represents relative positions (Figure 4).

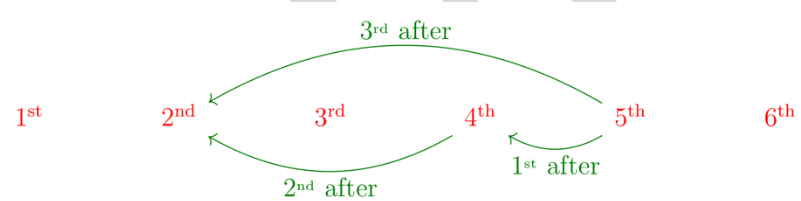

Figure 4 - Positions and Relative Positions

Furthermore, we have two types of additions: one within the addition system (Figure 5) and another one between the two subsystems (Figure 6), where we reach from one position to another by adding a relative position.

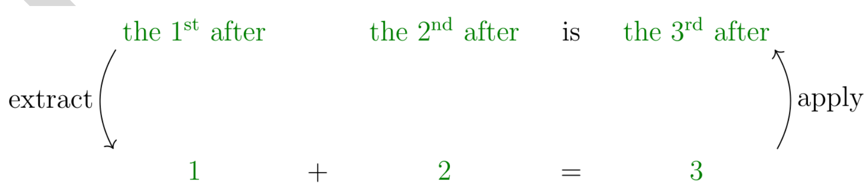

Figure 5 - Addition Within the Addition System of Relative Positions

$$\begin{array}{ccccccc} & \text{the } 3^{\text{rd}} \text{ after} & & \text{the } 2^{\text{nd}} & \text{is the} & \text{the } 5^{\text{th}} & \\ \text{extract} \downarrow & & & & & & \uparrow \text{apply} \\ & 3 & + & 2 & = & 5 & \end{array}$$

Figure 6 - Positions (Represented by the Succession System)

Added by Relative Positions (Represented by the Addition System)

Similarly, when using multiplication to count collections of apples, we have a pair of subsystems: an addition system for counting apples in a collection and a system of natural numbers for counting collections (Figure 7 and 8).

$$1 \text{ apple} \quad 2 \text{ apples} \quad 3 \text{ apples} \quad \cdots \xrightarrow{4 \text{ collections}} 11 \text{ apples} \quad 12 \text{ apples} \quad 13 \text{ apples}$$

$$\begin{array}{ccccccc} & 4 \text{ collections} & \text{, each containing} & 3 \text{ apples} & \text{, makes} & 12 \text{ apples} & \\ \text{extract} \downarrow & & & & & & \uparrow \text{apply} \\ & 4 & \times & 3 & = & 12 & \end{array}$$

Figure 7 - Apples (Represented by the Addition System)

Multiplied by Collections (Represented by the Natural Numbers)

$$\begin{array}{ccccccc} & 2 \text{ collections} & \text{, each containing} & 3 \text{ collections} & \text{, makes} & 6 \text{ collections} & \\ \text{extract} \downarrow & & & & & & \uparrow \text{apply} \\ & 2 & \times & 3 & = & 6 & \end{array}$$

Figure 8 - Multiplication of Collections in the Natural Numbers

There are two important notes to take away here. Firstly, an extraction can generally involve multiple number systems (and possibly multiple copies of the same system) with interactions among them. Secondly, the successor system, the addition system, and the natural numbers are related but distinct systems. For example, there is a number called "five" in each of these systems. We can draw analogies between these numbers that happen to share the same name, but they are ultimately different numbers as they express different meanings.

5. Systems for Proportions

Another well-known system is the real numbers. The prototypical way of extracting the real numbers is to consider the notion of proportion. Let us suppose that we have a train station with a straight train track that extends indefinitely to the west and to the east, imagining hypothetically that the Earth is flat. Firstly, we shall formalise the concept of position (Figure 9).

- "0" represents where the track meets the station.
- "1km E" represents the position on the track that is 1 kilometre east of the station.
- "4.276km E" represents the position on the track that is 4.276 kilometres east of the station.
- "2km W" represents the position on the track that is 2 kilometres west of the station.

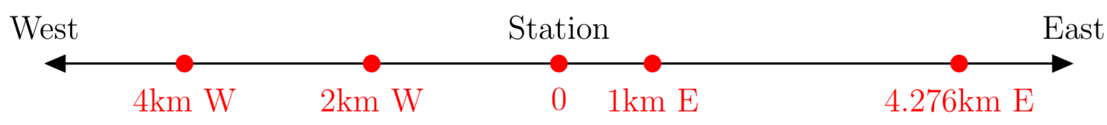

Figure 9 – Positions on a Train Track

We would then extract a system of numbers permitting comparisons. For example, 2km W is to the east of 4km W.

However, we do not have features like addition or multiplication. Indeed, it does not mean much to "add" positions: what position would the sum of Hong Kong and Oxford be? We can then formalise the concept of movements.

- "0" represents "staying still".
- "3km E" represents "moving 3 kilometres eastward".
- "0.889km W" represents "moving 0.889 kilometres westward".

We now arrive at a system where we can compare and add (Figure 10), though we are unable to multiply.

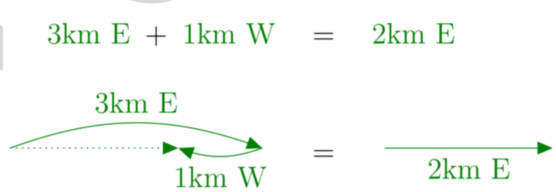

Figure 10 – Additions of Movements

We can also move from one position to another by "adding" a movement (Figure 11).

$$\begin{aligned} 2\text{km W} + 2\text{km W} &= 4\text{km W} \\ 3\text{km E} + 2\text{km W} &= 1\text{km E} \end{aligned}$$

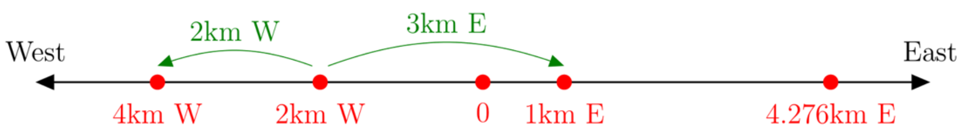

Figure 11 - Positions Added by Movements

Finally, we can formalise the concept of proportions.

- ❖ "4" represents "scaling a movement by a factor of 4".
- ❖ "-0.5" represents "scaling a movement by a factor of 0.5 and switching direction".
- ❖ "0" represents "scaling to no movement".

As expected, we can scale a movement by "multiplying" with a proportion (Figure 12).

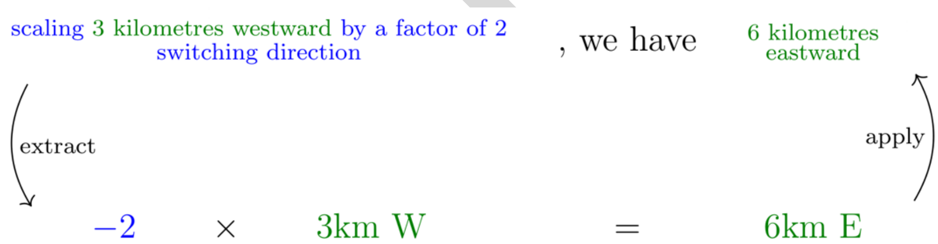

Figure 12 – Movements Scaled by Factors

We have a system representing proportions where we can add, subtract, multiply and divide (Figure 13).

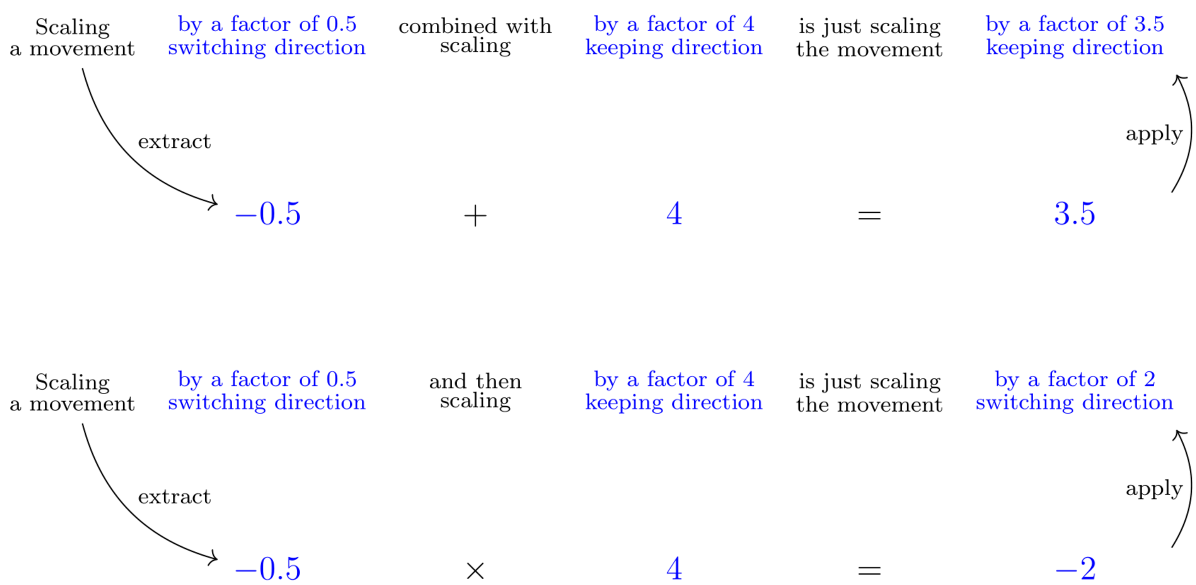

Figure 13 – Mathematical Operations on Scales

The concept of proportion comes up in many situations besides our train track example. From architecture to music, from sociology to physics, proportions are everywhere. The system of real numbers is indeed the standardised system extracted from proportions. However, in order to reach the real numbers, we have to make a variety of assumptions and decisions. This is where we need to conduct concept- and system-oriented analysis. This results in a lot of technical details, which I am glossing over.

One question does merit particular attention, however. Should we include infinitesimal proportions? In the case of a train track, to have infinitesimal proportions is to have small undetectable movements. From the perspective of measurements, there is no need for minuscule movements, as such things will never be recorded; this is the view held by the system of real numbers, which has no infinitesimals. However, they formalise the intuition of a very small object or movement and are useful for describing things from tiny superheroes to calculus.

6. Representation of Numbers

Good representations of numbers are crucial for their applicability. Here, we will focus on the representation of real numbers. Natural numbers are generally represented as a subsystem of the reals. Numbers have been represented in many ways in various civilisations and in various mediums (from carvings to electrical charges). However, aside from variations in syntax, the place–value notation has become the world standard.

- ❖ The place–value notation requires a chosen natural number greater than one as the base. Ten is the commonly-chosen base, and we will use it here for illustration.
- ❖ There are ten digits (such as "0", ..., "9" in Arabic numerals) to represent the numbers zero through nine along with additional symbols (such as "-", "." and ",") for indicating negativity and position.
- ❖ In the representation of a number, digits are put in a sequence. Each position, based on where it appears in the sequence, is given a distinct weight that is a power of ten. A negative sign indicates that the number is negative. (Figure 14)

$$-80.67 \quad = \quad -1 \times \left(8 \times 10^{1} + 0 \times 10^{0} + 6 \times 10^{-1} + 7 \times 10^{-2}\right)$$

Figure 14 – Place-Value Notation for Real Numbers

- ❖ The sequence of digits may be infinite and symbols are used to indicate that the sequence is endless or repeats indefinitely (Figure 15).

$$\pi = 3.141592\ldots \qquad \frac{2}{11} = 0.181818\ldots = 0.\overline{18}$$

Figure 15 – Numbers with Endless Digits

We will consider and discuss some characteristics of this notation.

- It allows for a concise and precise representation of a large range of numbers. For comparison, consider the representation of natural numbers "one-by-one" using fingers, apples, grains of sand, etc.:
    - One would have trouble representing large numbers, such as 10000, with fingers or apples.
    - A few grains of sand can represent a small number while a heap can represent a large number. However, moving and counting sand grains is tedious: it would be tricky to preform precise calculations and comparisons.
- Many important numbers, such as π, would require infinitely many irregular digits and so they could only be approximated in this notation. In most practical instances, approximations of these numbers are quite sufficient. The place–value notation can simply be extended (with notations like "π") when exact computations are required.
- The scale of numbers is made transparent and it is easy to compare numbers that are close by. For example, consider these numbers and their representations (Figure 16).

$$\begin{aligned} 2^{16} &= 65536 \\ \frac{22}{7} &= 3.142857\ldots \\ \pi &= 3.141592\ldots \end{aligned}$$

Figure 16 – Numbers in Various Representations

From the right-hand side, it is clear that the first number is at a much larger scale than the second, which is just a little bigger than the third. In contrast, this information is not laid out on the left.

- It only requires a fixed number of specified symbols which is useful for memorisation and implementation in computers. In contrast, the words "ten", "hundred", "thousand", etc. do not follow an easy pattern.
- Each digit is a symbol and is used the same way in all positions. In contrast, Roman numerals require multiple and distinct symbols to represent the same value at different positions. For example, "VI", "LX", "DC" represent "6", "60", "600" respectively.
- It allows for effective computations. The commonly-taught method of vertical calculation takes advantage of various mathematical tricks that are made applicable by this representation. Furthermore, these computational methods only require a fixed amount of memorisation (e.g. the multiplication table), which is again useful for computers. For example, we can see below how summing 67 with 324 can be done neatly with the standard method, saving a lot of tedious steps (Figure 17).

$$
\begin{array}{r}
{\scriptstyle 1} \\
6\;7 \\
+\;3\;2\;4 \\
\hline
3\;9\;1
\end{array}
\qquad
\begin{aligned}
& 67 + 324 \\
={}& (6 \times 10 \;+\; 7) \;+\; (3 \times 100 \;+\; 2 \times 10 \;+\; 4) \\
={}& (3 \times 100) \;+\; (6 \times 10 + 2 \times 10) \;+\; (7 + 4) \\
={}& (3) \times 100 \;+\; (6 + 2) \times 10 \;+\; (7 + 4) \\
={}& 3 \times 100 \;+\; 8 \times 10 \;+\; (1 \times 10 \;+\; 1) \\
={}& 3 \times 100 \;+\; 9 \times 10 \;+\; 1 \\
={}& 391
\end{aligned}
$$

Figure 17 – An Addition Method Using Various Tricks

- Numbers in weaker subsystems (such as the succession system and the addition system) are represented in the exact same way. These differences are somewhat respected in languages (e.g. "third", "three" and "thrice"), but they are not indicated in mathematical computations. Introducing and using notations (such as "1st") for distinguishing numbers in different systems would potentially improve understanding and reduce errors in reasoning.
- It requires a choice of a base. Various bases have different uses. There are a few factors to consider.
    - How intuitive is the base? This is relevant for humans but not so much for computers.
    - How large or small is the base? Smaller bases have less memorisation and easier computations, at the cost of longer representations. For example, the multiplication table in base-two only has three entries (0×0=0, 1×0=0 and 1×1=1) while the representation for a thousand is long (1111101000).
    - How divisible is the base? When trying to share the bill for a meal evenly, three people often run into issues. This is essentially because the common monetary systems use base-ten, in which one third has an infinite number of digits (0.333...). In contrast, ten is a multiple of five, so base-ten is good

for divisions by five. A base with many small factors would be desirable: divisions of natural numbers are less likely to require infinitely many digits and multiplication tables are easier to memorise.

Modern computers, which speak the language of on-and-off signals, are most suitable for a base-two notation. However, there might be a "chicken and egg" dilemma here: base-two computations are easy to implement, and this would conversely promote the use of on-and-off signals in computers.

Which base is best for humans? Mediums (such as text and speech) used by humans have room for more digits, allowing for a larger base. At the same time, we are not terribly good at large amounts of precise memorisation and a base too large might be difficult to use. Ten is probably a decent size and is quite intuitive since each of us typically has ten fingers. However, it has the drawback of not being divisible by three. There are many common situations in which one needs to divide by three, so bases like six and twelve might be better alternatives. Regardless, it would be a tremendous task to have societies around the world switch to a different base, and the benefits might be insignificant, as computers are gradually performing more computations for us. I shall leave the rest of this debate for the reader.

Representations of numerical operations are important as well. We will look at two factors in particular.

- To express the sum of two and three, we put a plus sign in-between the represented numbers: "2+3". This is intuitive as the plus sign connects the two numbers and indicates that we should combine them. Nonetheless, other conventions are available. For example, the Polish notation puts the plus sign in front of the numbers (e.g. + 2 3) and this can be more efficient for computers.
- Brackets are used to indicate the order of operations in an expression. To avoid the clustering of brackets, it is helpful to have operation precedents. For example, the common standard is that multiplications are done before additions unless bracketed otherwise (Figure 18).

The expression $2\times3+5$

would mean $(2\times3)+5$

and not $2\times(3+5)$.

Figure 18 – Operation Precedence

This is a good convention because we can expand any complicated expression containing multiplications and additions into one that does not need any brackets (Figure 19).

$((a+b) \times (c+d)) + e$ can be rewritten as

$a\times c + a\times d + b\times c + b\times d + e$ which is bracket-free.

Figure 19 - Brackets Removed by Expansions

In contrast, this is not the case if we take the convention of doing additions before multiplication. For example, it would not be possible to remove the bracket in the expression (a×b)+c.

Overall, we see that there are a lot of factors that contribute to the design of representations for number systems. Different notations are suited to different purposes (e.g. approximate vs. exact) and users (e.g. humans vs. computers). The current standard of notation, though not without its flaws, works well and will undoubtedly continue to be used to represent the real and natural numbers in the foreseeable future.

7. Conclusion

Counting and proportions underpin a lot of concepts. It is therefore not surprising that their standard number systems -- the natural numbers and the real numbers -- have been icons of mathematics. However, it would be fallacious to use these numbers blindly. Many subtle assumptions and decisions are made in a formalisation, and the resulting system can involve a range of interacting subsystems. Alternative number systems are relevant in many situations. Regardless, the natural and real numbers are very flexible, and other systems can often be built from them. They are certainly well-rounded, applicable tools to have, though please feel free to try to improve upon them.

Reviews for ‘The Design of Everyday Numbers’
by Matthew Lau (STAAR 9 -2019)

Review 1 – Keith Lau – Accept

1. Is the subject matter of the article suitable for an interdisciplinary audience?: Yes. The content may seem simple, but the underlying ideas can be applied to other fields.

2. Does the title reflect the subject matter of the article?: Yes.

3. Does the article make a contribution to the discussion in its field?: I am not certain about my answer here. I do not know how mathematicians (with higher training than myself) would receive this article. For me, I am more particular to how I might use this content in discussion with my students (grades 9-12). Most of my students just view math as a class they have to take, but of course, some do like math and wish to pursue it in college. Would the ideas of this article help them appreciate math (or numbers) more? I don't know. And how would I break down the ideas of this article to them in way they would understand and appreciate?

4. Is the article clearly written?: Yes. However, I feel that the definitions of system-oriented and concept-oriented approaches (section 2, Comparing Apples and Oranges) can be a little confusing or hard to understand for some.

5. Is the article well structured?: Yes. I am not sure what I could recommend.

6. Are the references relevant and satisfactory?: I did not see any references.

7. Do you feel the article appropriately uses figures, tables and appendices?: Yes.

8. What is your recommendation?: Accept

Reviewer's comments to the author (this will be made public on acceptance of the article):

Review 2 - Martin Ku - Minor

1. Is the subject matter of the article suitable for an interdisciplinary audience?:
Yes. The subject matter is suitable for audience with basic prior knowledge of numbers and arithmetic. People with background in mathematics and mathematical education especially would find it interesting.

2. Does the title reflect the subject matter of the article?: Yes. The title is a good general description of the core ideas discussed in the article.

3. Does the article make a contribution to the discussion in its field?: Yes. The article gives insights to people without the background of number theory and abstract algebra so that they can look at the fundamental properties of numbers both intuitively and systematically.

4. Is the article clearly written?: The article is basically clearly written. However, there are some parts of the article that can be clearer, especially for people without strong mathematical background:

A. In the part that introduces the system-oriented direction and the concept-oriented direction of analysis (line 59 to line 126), there are lots of information about these two directions. However, more explanations of these two directions are needed. Since the two directions are used as the conceptual frameworks for the discussion thereafter, the validity of these two directions is very critical. There are some questions that may need to be addressed. Why should we look at the matter discussed in the article in these two particular directions? How are these two directions different from each other?

B. In the discussion about the representation of numbers in section 6 (line 375 to line 513), it is not clear what conceptual frameworks are used for the analysis. Quite a number of characteristics of the representation of real number are discussed, but it is difficult to understand these characteristics systematically, as there is no apparent organization of these characteristics. More prominent organization and proper categorization of these characteristics may be necessary for the audience to understand the ideas, as well as to see the relevance of the characteristics mentioned in this section.

5. Is the article well structured?: The article is basically well-structured in a way that the audience can understand the discussion in a logical manner. However, the structures of some parts can be improved:

A. When discussing the system-oriented direction and the concept-oriented direction (line 59 to 126), the system-oriented direction is discussed first, and the concept-oriented direction is discussed later. The discussions for both of them are quite extensive. However, when applying these two conceptual frameworks in section 3 of the article (line 128 to line 251), the analysis with the concept-oriented framework comes first, and the analysis with the system-oriented framework is very brief. This may cause confusion among the audience. Some explanations should be added to justify the ways of using these two conceptual frameworks for the analysis.

B. Sections 3 to 5 can be articulated to section 2 more clearly. Only a fraction of the results from the analysis in section 2 have been used in sections 3, 4 and 5, and the relationships are not stated explicitly. The audience would want to know why they should focus on these particular aspects. It would be easier for the audience to follow the logic if the relevant parts of the conceptual frameworks are indicated when examining the natural number system and the real number system.

6. Are the references relevant and satisfactory?: When discussing the system-oriented direction and the concept-oriented direction (line 59 to 126), relevant references should be added to back up the claims. The system-oriented and concept-oriented directions are fundamental to the discussions thereafter. Readers would wonder if the descriptions of these two approaches are complete, and why these two directions are appropriate for understanding the design of everyday numbers.

In general, it is uncommon for a journal article in mathematics to have as many references as the counterparts in other fields like social science. However, for an article that is targeting general audience, more references may be needed to support the qualitative descriptions. For instance, the descriptions about the representation of real numbers are mostly qualitative, and it would be better to have some references to indicate that the characteristics described are accurate and relevant.

7. Do you feel the article appropriately uses figures, tables and appendices?: Yes. The figures in the article illustrate the ideas very clearly.

8. What is your recommendation?: Minor revision

Reviewer's comments to the author (this will be made public on acceptance of the article): The article gives the audience without a strong mathematical background a glimpse of how we can approach numbers in daily-life both intuitively and systematically. This is a good opportunity for the general audience to understand mathematical knowledge that goes beyond mechanical mathematical operations. With the insights from the article, the general audience can start articulating the concepts they have learnt at school and the experiences of using numbers in the daily-life.